Non-archimedean Quantum K-theory and Gromov-Witten Invariants

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Plan: 1. Motivations from mirror symmetry
2. Review of derived non-archimedean geometry
3. Representability theorem
4. Moduli stack of non-archimedean stable maps and Gromov compactness
5. Numerical enumerative invariants
6. Geometric relations between the derived moduli stacks
Def: A smooth projective variety $X/C$ is called Calabi-Yau if its canonical bundle $K_X$ is trivial, i.e. it has a nowhere vanishing holomorphic volume form.

Examples: Elliptic curve, abelian variety, $K3$ surface, hypersurface of degree $d+1$ in $CP^d$.

Mirror Symmetry: conjectural duality between Calabi-Yau varieties:

Any CY variety $X \leftrightarrow \exists$ mirror variety $\tilde{X}$

such that a list of deep geometric relations hold between $X$ and $\tilde{X}$, involving: Hodge structures, Gromov-Witten invariants, Fukaya categories, derived category of coherent sheaves, SYZ torus fibrations, etc.

A more careful study $\rightarrow$ mirror symmetry is not really a duality between individual CY varieties, but rather a duality between “maximally degenerating families” of CY varieties.

Example: Type III degeneration of $K3$ surfaces
In general, an algebraic family of varieties over a punctured disk \( \mathbb{C}^* \) field of formal Laurent series.

\[
\text{non-archimedean field: norm } |x| = e^{-\text{val}} \\
|x+y| \leq \max\{|x|, |y|\}
\]

Non-archimedean geometry: analog of complex geometry over non-archimedean fields.

More general, more symmetric formulation of mirror symmetry as a duality of non-archimedean Calabi-Yau manifolds (with maximal degenerations)

Advantages:
1) Working formally without worrying about complex analytic convergence.
2) Existence of SYZ torus fibration is proved (Nicaise - Xu - Yu 2019)
3) New ways for counting curves with boundaries \( \rightarrow \) wall-crossing formulas

These considerations motivate an analog of Gromov-Witten theory in non-archimedean geometry.

Classical approach to Gromov-Witten theory: Perfect obstruction theory by Behrend-Fantechi, Li-Tian

Our approach in the non-archimedean setting: We develop a theory of derived non-archimedean geometry \( \rightarrow \) non-archimedean quantum K-invariants \( \rightarrow \) non-archimedean Gromov-Witten invariants

2. Review of derived non-archimedean geometry
Q: What is a derived non-archimedean analytic space?
Recall the definition of a derived scheme:
A derived scheme is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf $\mathcal{O}_X$ of simplicial commutative rings on $X$, satisfying the following conditions:

1. The ringed space $(X, \pi_0(\mathcal{O}_X))$ is a scheme.
2. For each $j \geq 0$, the sheaf $\pi_j(\mathcal{O}_X)$ is a quasi-coherent sheaf of $\pi_0(\mathcal{O}_X)$-modules.

In order to adapt the above definition to (non-archimedean) analytic geometry, we need to find a way to impose additional analytic structures on the sheaf $\mathcal{O}_X$, e.g. • a notion of norms on the sections of $\mathcal{O}_X$
  • compose the sections of $\mathcal{O}_X$ with convergent power series

Our first attempt: Enhance simplicial commutative rings with non-archimedean analytic structures. “simplicial commutative affinoid/Banach algebras”
Difficult: Banach structure and simplicial structure do not mix well.
  (works by Ben-Bassat, Kremnizer, Bambozzi, …)

Our strategy: Use the theory of pregeometry and structured topos of Lurie.

Idea: Use the language of $\infty$-category/$\infty$-topos to generate derived sheaves starting from simple classical objects, bypassing any model-dependent constructions (e.g. simplicial algebras, dg-algebras).
Def: A pregeometry is a category $T$ equipped with a class of admissible morphisms and a Grothendieck topology generated by admissible morphisms, such that

1. $T$ admits finite products
2. The class of admissible morphisms is closed under composition, pullback and retract.
3. $g, h$ admissible $\Rightarrow f$ admissible

Examples: $k$ non-archimedean field
- $\text{Jet}(k)$ := category of smooth $k$-varieties, étale maps, étale topology
- $\text{Jan}(k)$ := category of smooth $k$-analytic spaces, étale maps, étale topology

Def: $T$ pregeometry, $X$ $\infty$-topos (e.g. the category of sheaves of spaces on a given topological space)
A $T$-structure on $X$ is a functor $O: T \to X$ s.t.
1. it preserves finite products.
2. it sends pullbacks of admissible morphisms in $T$ to pullbacks in $X$.
3. it sends coverings in $T$ to effective epimorphisms in $X$.

The idea behind this abstract definition:
We can think of a $\text{Jan}(k)$-structure $O$ as a sheaf of derived rings equipped with an analytic structure:
(1) Let $F := \mathcal{O}(\mathbb{A}') \in \mathcal{X}$

sum $+: \mathbb{A}' \times \mathbb{A}' \to \mathbb{A}' \xrightarrow{\text{product-preserving}} +: \mathcal{F} \times \mathcal{F} \to \mathcal{F}$
multiplication $\cdot : \mathbb{A}' \times \mathbb{A}' \to \mathbb{A}' \xrightarrow{\text{product-preserving}} \cdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$

Therefore, we can intuitively think of $F$ as a sheaf of simplicial commutative rings.

(2) The sheaf $F$ is also equipped with norms:
Let $\mathbb{D}' \subset \mathbb{A}'$ closed unit disk

Recall: $\mathcal{O}$ sends pullbacks of admissible morphisms in $\mathcal{T}$ to pullbacks in $\mathcal{X}$

$\implies \mathcal{O}(\mathbb{D}') \hookrightarrow \mathcal{F}$ is a monomorphism

Therefore, we can think of $\mathcal{O}(\mathbb{D}')$ as the subsheaf of $F$ consisting of functions of norm $\leq 1$.

(3) A convergent power series $f$ on $\mathbb{D}' \xrightarrow{\text{functoriality}}$ morphism $\mathcal{O}(\mathbb{D}') \to \mathcal{F}$

We think of it as composition with $f$

Now we are ready to give the definition of derived non-archimedean analytic space.

Def: A derived $k$-analytic space $X$ is a pair $(X, \mathcal{O}_X)$ consisting of a (hypercomplete) ∞-topos $\mathcal{X}$ and a $\mathcal{T}_{an}(k)$-structure $\mathcal{O}_X$ on $X$ s.t.

(1) $(X, \pi_0(\mathcal{O}_X^{alg}))$ is equivalent to the ringed ∞-topos associated to the étale site of a $k$-analytic space.

(2) For every $j \geq 0$, $\pi_j(\mathcal{O}_X^{alg})$ is a coherent sheaf of $\pi_0(\mathcal{O}_X^{alg})$-modules.
2. Representability theorem

Q: How do derived analytic spaces appear in nature?
A: Via the representability theorem

Representability theorem (Porta-Y):
Let $F$ be an analytic moduli functor (i.e. a sheaf over the \'{e}tale site of derived analytic spaces). The followings are equivalent:
1) $F$ has the structure of a derived analytic space
2) $F$ is compatible with Postnikov towers, has an analytic cotangent complex, and its truncation is an analytic space.

Rem: The representability theorem has two important implications:
1) Philosophical: Our notion of derived analytic space is natural and sufficiently general. I.e. any reasonable analytic moduli functor has the structure of a derived analytic space.
2) Practical: The conditions are easy to verify in practice. So the theorem gives plenty of down-to-earth examples of derived analytic spaces.

Rem: We say that a moduli functor $F$ is compatible with Postnikov towers if it is infinitesimally cohesive and nilcomplete.
- infinitesimally cohesive: $F$ sends square-zero extensions to pullbacks
- nilcomplete: $F(X) \rightsquigarrow F(\text{tens} X)$
Rem: We also proved a generalization of the representability theorem for non-archimedean analytic stacks.

3. Moduli stack of non-archimedean stable maps and Gromov compactness

Fix $X$ a smooth rigid $k$-analytic space.

Def: Let $T$ be a derived $k$-analytic space. An $n$-pointed genus $g$ stable map into $X$ over $T$ consists of an $n$-pointed genus $g$ prestable curve $[C \to T, (s_i)]$ over $T$ and a map $f: C \to X$, s.t. every geometric fiber $[C_t, (s_i(t)), f_t: C_t \to X]$ is a stable map, in the sense that its automorphism group is a finite analytic group.

Representability theorem $\rightarrow$ derived enhancement of the moduli stack of non-archimedean analytic stable maps

Theorem (Porta-Y): The derived moduli stack $\mathcal{IRM}_{g,n}(X)$ of $n$-pointed genus $g$ stable maps into $X$ is representable by a derived $k$-analytic stack locally of finite presentation and derived lci.

cotangent complex is perfect and in tor-amplitude $[1, -\infty)$
Theorem (Non-archimedean Gromov compactness, $Y$): Assume further more that $X$ is proper and equipped with a Kähler structure. Given any curve class $\beta$, the substack $\overline{\text{M}_g,n}(X,\beta) \subset \overline{\text{M}_g,n}(X)$ is a proper $k$-analytic stack, hence $\text{IR}\overline{\text{M}_g,n}(X,\beta)$ is a proper derived $k$-analytic stack.

4. Numerical enumerative invariants

Q: Given the compactness, how do we obtain numerical enumerative invariants from the derived structure?

Two ways classically:

\[ K\text{-theory} \quad \text{---> quantum } K\text{-invariants (Givental-Lee)} \]
\[ \text{intersection theory} \quad \text{---> Gromov-Witten invariants} \]
\[ \text{(Behrend-Fantechi)} \]

K-theory works similarly in non-archimedean geometry:

Def: The non-archimedean quantum $K$-invariants are the maps

\[
K^X_{g,n,\beta} : K_0(X)^{\otimes n} \rightarrow K_0(\overline{\text{M}_g,n})
\]
\[
a_1 \otimes \cdots \otimes a_n \mapsto st_* (ev_1^* a_1 \otimes \cdots \otimes ev_n^* a_n)
\]

where

\[
\text{IR}\overline{\text{M}_g,n}(X,\beta) \quad \text{ev}_i \rightarrow \quad X
\]

stabilization of domain
However, intersection theory (in the sense of Fulton's book) does not work in non-archimedean geometry (nor complex analytic geometry). Reason: there are not enough cycles to have moving lemma, or to have Chern classes from vector bundles.

Solution: work with cohomological theories.

Choices: 1. Étale cohomology $\rightarrow$ invariants in $\mathbb{Q}_\ell$, independence of $\ell$
2. de Rham cohomology $\rightarrow$ invariants in $k$, still not ideal
3. Berkovich integral étale cohomology $\rightarrow$ invariants in $\mathbb{Q}$
   functorial properties not sufficiently developed
   only works over $\mathbb{C}((t))$.
4. Rigid analytic motivic cohomology by Ayoub $\rightarrow$ invariants in $\mathbb{Q}$
   works over general non-archimedean fields
   six functor formalism recently developed by Ayoub–Gallauer–Vezzani

Roughly, any $k$-analytic space $X$

$\rightarrow$ $\text{RigSH}_\text{ét}(S; \mathbb{Q})$ $\infty$-category of étale $k$-analytic motives over $X$ with rational coefficients.

Six functors $\otimes, \text{Hom}, f^*, f_*, f!, f^!$

For any $a: X \to S$ $k$-analytic space/stack, we have

Motivic cohomology: $H^b(X/S, \mathbb{Q}(r)) = \text{Hom}_{\text{RigSH}_\text{ét}(S; \mathbb{Q})}(1_S, a^* a^* \mathbb{Q}(r)[q])$

Motivic Borel–Moore homology:

$H^m_\mathbb{Z}(X/S, \mathbb{Q}(r)) := \text{Hom}_{\text{RigSH}_\text{ét}(S; \mathbb{Q})}(1_S(r)[q], a_* a^! \mathbb{Q}), \quad q, r \in \mathbb{Z}.$
Next we apply a derived analog of deformation to the normal cone following Khan:

Theorem (Khan-Rydh): For any derived lci morphism $f: X \to Y$ of derived $k$-analytic stacks, there exists a derived lci derived $k$-analytic stack $D_{X/Y}$ over $Y \times \mathbb{A}^1$, and a derived lci morphism $X \times \mathbb{A}^1 \to D_{X/Y}$ over $Y \times \mathbb{A}^1$, whose fiber over $G_m = \mathbb{A}^1 \setminus 0$ is $X \times G_m \to Y \times G_m$, and the fiber over $0 \in \mathbb{A}^1$ is the $0$-section $X \to T_{X/Y}[1]$ to the shifted tangent bundle.

\[ d := \text{virtual dim of } X \to Y \]

Def (Khan): The virtual fundamental class of $f: X \to Y$ is the class

\[ [X/Y] := f^!(1) \in H^{BM}_{2d}(X/Y, \mathbb{Q}(d)) \quad \text{where} \quad 1 \in H^{BM}_0(Y/Y, \mathbb{Q}). \]

Def: The non-archimedean Gromov-Witten invariants are the maps

\[ I_{g,n, \beta}^X: H^*(X, \mathbb{Q}(*))^\otimes n \to H^*(\overline{M}_{g,n}, \mathbb{Q}(*)) \quad \text{where} \quad a_1 \otimes \cdots \otimes a_n \to \text{PD}^{-1} st_x((\text{ev}_1^*a_1 \cdots \otimes \text{ev}_n^*a_n) \cap [\overline{M}_{g,n}(X, \beta)]) \]

where

\[ \overline{M}_{g,n}(X, \beta) \xrightarrow{\text{st}} \overline{M}_{g,n} \]

stabilization of domain
6. Geometric relations between the derived moduli stacks

Next we need to establish all the expected properties of our non-archimedean invariants. They will follow readily from a list of natural geometric relations between the derived moduli stacks.

In order to state the geometric relations, instead of working with \( n \)-pointed genus \( g \) stable maps, we use a slight combinatorial refinement called \((\tau, \beta)\)-marked stable maps for any \(A\)-graph \((\tau, \beta)\) introduced by Behrend-Manin. (It imposes degeneration types on the domains of stable maps as well as more refined curve classes.)

\[ \xymatrix{ \text{associated moduli stacks } \overline{\mathcal{M}}(X, \tau, \beta) \text{ of } (\tau, \beta)\text{-marked stable maps,} \\
\text{and their derived enhancements } \mathcal{R}\overline{\mathcal{M}}(X, \tau, \beta) } \]

Furthermore, it will be useful to consider the relative situation \( \mathcal{R}\overline{\mathcal{M}}(X/S, \tau, \beta) \)

**Theorem (Relations of derived moduli stacks, Porta-Y):**

Let \( S \) be a rigid \( k \)-analytic space and \( X \) a rigid \( k \)-analytic space smooth over \( S \). The derived moduli stack \( \mathcal{R}\overline{\mathcal{M}}(X/S, \tau, \beta) \) of \((\tau, \beta)\)-marked stable maps into \( X/S \) satisfies the following geometric relations with respect to elementary operations on \(A\)-graphs:

1) **Products:** \((\tau_1, \beta_1), (\tau_2, \beta_2)\) \(A\)-graphs

\[ \mathcal{R}\overline{\mathcal{M}}(X/S, \tau_1 \cup \tau_2, \beta_1 \cup \beta_2) \xrightarrow{\sim} \mathcal{R}\overline{\mathcal{M}}(X/S, \tau_1, \beta_1) \times_S \mathcal{R}\overline{\mathcal{M}}(X/S, \tau_2, \beta_2) \]
2) Cutting edges: \((\tau, \beta) \rightarrow (\sigma, \beta)\)

\[
\begin{array}{c}
\text{cut} \\
& \downarrow
\end{array}
\]

We have a derived pullback diagram

\[
\begin{array}{ccc}
\text{IRM}(X/S, \tau, \beta) & \longrightarrow & \text{IRM}(X/S, \sigma, \beta) \\
\downarrow \text{ev}_e & & \downarrow \text{ev}_i \times \text{ev}_f \\
X & \Delta & X \times_S X
\end{array}
\]

3) Universal curve: \((\tau, \beta) \rightarrow (\sigma, \beta)\)

\[
\begin{array}{c}
\text{forget the tail } t \\
& \downarrow
\end{array}
\]

We have a derived pullback diagram

\[
\begin{array}{ccc}
\text{IRM}(X/S, \tau, \beta) & \longrightarrow & \text{IRM}(X/S, \sigma, \beta) \\
\downarrow \text{C}^{\text{pre}}_w & & \downarrow \text{M}^{\text{pre}}_\sigma \\
\text{universal curve corresponding to } w
\end{array}
\]

4) Forgetting tails: Context as above, we have a derived pullback diagram

\[
\begin{array}{ccc}
\text{IRM}(X/S, \tau, \beta) & \longrightarrow & \text{M}_c \times \text{IRM}(X/S, \sigma, \beta) \\
\downarrow \text{C}^{\text{pre}}_w & & \downarrow \text{M}_c \times \text{M}^{\text{pre}}_\sigma \\
\end{array}
\]
5) Contracting edges: \((\tau, \beta) \rightarrow (\sigma, \beta)\)

\[
\begin{array}{c}
\text{contract e} \\
\hline
\text{add k tails}
\end{array}
\]

\[
\begin{array}{c}
(\tau^i, \beta^i_j) \leftarrow (\sigma, \beta)
\end{array}
\]

\[
\begin{array}{c}
\text{expand e by a chain of length } \ell
\end{array}
\]

We have a natural equivalence

\[
\colim_{\ell \in \mathbb{N}} \mathcal{R}_\ell(X/S, \tau_i, \beta^i_j) \sim \overline{M}_\tau \times \mathcal{R}(X/S, \sigma, \beta)
\]

Rem: The universal curve relation in the particular case where \(\tau\) is a point:
The forgetful map \(\mathcal{R}_g, n+1(X/S) \rightarrow \mathcal{R}_g, n(X/S)\) is equivalent to the universal curve \(\overline{R}_g, n(X/S) \rightarrow \overline{R}_g, n(X/S)\).

Such an intuitive statement in fact incorporates all the information about virtual counts with respect to forgetting a tail, which is classically expressed and proved in terms of pullback properties of perfect obstruction theories and intrinsic normal cones.

Rem: We take further advantage of the flexibility of our derived approach to introduce a generalized type of Gromov-Witten invariants that allow not only simple incidence conditions for marked points, but also incidence conditions with multiplicities. They satisfy a list of properties parallel to Behrend-Manin axioms. To the best of our knowledge, such invariants are not yet considered in the literature, even in algebraic geometry.